

ON THE GENERATORS OF QUANTUM STOCHASTIC MASTER EQUATION.

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ABSTRACT. A characterisation of the stochastic bounded generators of quantum irreversible Master equations is given. This suggests the general form of quantum stochastic evolution with respect to the Poisson (jumps), Wiener (diffusion) or general Quantum Noise. The corresponding irreversible Heisenberg evolution in terms of stochastic completely positive (CP) maps is found and the general form of the stochastic completely dissipative (CD) operator equation is discovered.

1. INTRODUCTION

Recently a stochastic irreversible evolution has been introduced in quantum theory in order to describe the continuous measurements and random trajectories of individual quantum objects under their observation in time. In the quantum theory of open systems there is a well known Lindblad's form [19] of quantum irreversible master equation, satisfied by the one-parameter semigroup of completely positive (CP) maps. This is a nonstochastic equation, which can be obtained by averaging a stochastic Langevin equation over the driving quantum noises. On the other hand the Langevin equation is satisfied by a quantum stochastic process of dynamical representations, which are obviously completely positive due to *-multiplicativity of the representations. The representations give the examples of pure CP maps, but among pure CP maps there are not only the representations. This means a possibility to construct a pure irreversible quantum stochastic CP dynamics, which can not be driven by a Langevin equation.

The examples of such dynamics having recently been found many physical applications, will be considered in the first section. The rest of the paper will be devoted to the mathematical derivation of the general structure for the quantum stochastic evolution equations with the bounded coefficients. Here in the introduction we would like to outline this structure on the formal level.

In order to achieve this goal, we will extend the Evans-Lewis differential dilation theorem [17] for the generators of CP dynamics, to the stochastic differentials, generating an Itô *-algebra

$$(1.1) \quad d\Lambda(a)^\dagger d\Lambda(a) = d\Lambda(a^*a), \quad \sum \lambda_i d\Lambda(a_i) = d\Lambda\left(\sum \lambda_i a_i\right), \quad d\Lambda(a)^\dagger = d\Lambda(a^*)$$

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with given mean values $\langle d\Lambda(t, a) \rangle = l(a) dt$, $a \in \mathfrak{a}$. Here \mathfrak{a} is a (finite-dimensional) noncommutative \star -algebra with a self-adjoint annihilator (death) $d = d^*$, $\mathfrak{a}d = 0$, corresponding to $dt = d\Lambda(t, d)$, and $l : \mathfrak{a} \rightarrow \mathbb{C}$ is a positive $l(a^*a) \geq 0$ linear functional, normalized as $l(d) = 1$, corresponding to the determinism $\langle dt \rangle = dt$. The functional l defines the GNS representation $a \mapsto \mathbf{a} = (a_\nu^\mu)_{\nu=+, \bullet}^{\mu=-, \bullet}$ of \mathfrak{a} in terms of the quadruples

$$(1.2) \quad a_\bullet^\bullet = j(a), \quad a_+^\bullet = k(a), \quad a_\bullet^- = k^*(a), \quad a_+^- = l(a),$$

where $j(a^*a) = j(a)^\dagger j(a)$ is the matrix representation $j(a)^\dagger k(a) = k(a^*a)$ on the space \mathcal{K} of the Kolmogorov decomposition $l(a^*a) = k(a)^\dagger k(a)$ into the dot-product $a_\bullet^- a_+^\bullet$ with a finite number of the components indexed by \bullet , and $k^*(a) = k(a^*)^\dagger$.

The quantum stochastic processes $t \in \mathbb{R}_+ \mapsto \Lambda(t, a)$, $a \in \mathfrak{a}$ with independent increments $d\Lambda(t, a) = \Lambda(t + dt, a) - \Lambda(t, a)$, forming an Itô \star -algebra, can be represented [20] in the Fock space \mathfrak{F} over the space of \mathcal{K} -valued square-integrable functions on \mathbb{R}_+ as $\Lambda_\nu^\mu(t, a_\nu^\mu) = a_\nu^\mu \Lambda_\mu^\nu(t)$. Here

$$(1.3) \quad a_\nu^\mu \Lambda_\mu^\nu(t) = a_\bullet^\bullet \Lambda_\bullet^\bullet(t) + a_+^\bullet \Lambda_+^\bullet(t) + a_\bullet^- \Lambda_\bullet^-(t) + a_+^- \Lambda_+^-(t),$$

is the canonical decomposition of Λ into the exchange Λ_\bullet^\bullet , creation Λ_+^\bullet , annihilation Λ_\bullet^- and preservation (time) $\Lambda_+^- = tI$ processes of quantum stochastic calculus [20], having the mean values $\langle \Lambda_\mu^\nu(t) \rangle = t \delta_+^\nu \delta_\mu^-$ with respect to the vacuum state in \mathfrak{F} . Thus the parametrising algebra \mathfrak{a} can be always identified with a \star -subalgebra of the algebra of all quadruples $\mathbf{a} = (a_\nu^\mu)_{\nu=+, \bullet}^{\mu=-, \bullet}$, where $a_\nu^\mu : \mathcal{K}_\nu \rightarrow \mathcal{K}_\mu$ are the linear operators on $\mathcal{K}_\bullet = \mathcal{K}$, $\mathcal{K}_+ = \mathbb{C} = \mathcal{K}_-$, having the adjoints $a_\nu^{\mu*} \mathcal{K}_\mu \subseteq \mathcal{K}_\nu$, with the Hudson–Parthasarathy (HP) multiplication table [16]

$$(1.4) \quad \mathbf{a} \bullet \mathbf{b} = (a_\nu^\mu b_\nu^\bullet)_{\nu=+, \bullet}^{\mu=-, \bullet},$$

the unique death $\mathbf{d} = (\delta_-^\mu \delta_\nu^+)_{\nu=+, \bullet}^{\mu=-, \bullet}$, and the involution $a_{-\nu}^{*\mu} = a_{-\mu}^\nu$, where $-(-) = +$, $-\bullet = \bullet$, $-(+) = -$.

The main result of this paper is the derivation of the general structure for the unbounded generators $\lambda_\nu^\mu : \mathcal{B} \rightarrow \mathcal{B}$ of the linear quantum stochastic CP evolutions ϕ_t over $\mathcal{B} = \mathcal{L}(\mathcal{H})$ in terms of the quantum stochastic differentials $d\phi = \phi \circ \lambda_\nu^\mu d\Lambda_\mu^\nu$ with $\phi_0 = \iota$ at $t = 0$, where $\iota(B) = B$ is the identical representation of \mathcal{B} . As in the bounded case and finite dimensional Itô algebra [?], the generator $\boldsymbol{\lambda} = (\lambda)_{\nu=+, \bullet}^{\mu=-, \bullet}$ can be written in the “Lindblad” form $\boldsymbol{\lambda}(B) = \mathbf{L}^\dagger j(B) \mathbf{L} - \mathbf{K}^\dagger B - B \mathbf{K}$, defining the quantum stochastic differential equation as

$$d\phi_t(B) + \phi_t(K^\dagger B + B K - \mathbf{L}^\dagger j(B) \mathbf{L}) dt = \phi_t(L^\bullet j(B) L_\bullet - B \otimes \delta_\bullet^\bullet) d\Lambda_\bullet^\bullet$$

$$(1.5) \quad + \phi_t(L^\bullet j(B) L - K^\bullet B) d\Lambda_+^\bullet + \phi_t(L^\dagger j(B) L_\bullet - B K_\bullet) d\Lambda_\bullet^-,$$

where j is an operator representation of \mathcal{B} , and δ_\bullet^\bullet is the identity operator in \mathcal{K} . Such an extension of Lindblad’s form for quantum stochastic generators was discovered recently in [15] even for a nonlinear case. We shall prove that this structure is necessary at least in the case of the bounded w^* -continuous generators on a von Neumann algebra \mathcal{B} . The existence of minimal CP solution which is constructed under certain continuity conditions proves that this structure is also sufficient for the CP property of any solution to this stochastic equation.

The Evans–Lewis case $\Lambda(t, a) = \alpha t I$ is described by the simplest one-dimensional Itô algebra $\mathfrak{a} = \mathbb{C}d$ with $l(a) = \alpha \in \mathbb{C}$ and the nilpotent multiplication $\alpha^* \alpha = 0$ corresponding to the non-stochastic (Newton) calculus $(dt)^2 = 0$ in $\mathcal{K} = 0$. The standard Wiener process $Q = \Lambda_-^\bullet + \Lambda_+^\bullet$ in Fock space is described by the second order nilpotent algebra \mathfrak{a} of pairs $a = (\alpha, \xi)$ with $d = (1, 0)$, $\xi \in \mathbb{C}$, represented by the quadruples $a_+^- = \alpha$, $a_-^- = \xi = a_+^\bullet$, $a_-^\bullet = 0$ in $\mathcal{K} = \mathbb{C}$, corresponding to $\Lambda(t, a) = \alpha t I + \xi Q(t)$. The unital \star -algebra \mathbb{C} with the usual multiplication $\zeta^* \zeta = |\zeta|^2$ can be embedded into the two-dimensional Itô algebra \mathfrak{a} of $a = (\alpha, \zeta)$, $\alpha = l(a)$, $\zeta \in \mathbb{C}$ as $a_-^\bullet = \zeta$, $a_+^\bullet = +i\zeta$, $a_-^- = -i\zeta$, $a_+^- = \zeta$. It corresponds to $\Lambda(t, a) = \alpha t I + \zeta P(t)$, where $P = \Lambda_-^\bullet + i(\Lambda_+^\bullet - \Lambda_-^\bullet)$ is the representation of the standard Poisson process, compensated by its mean value t . Thus our results are applicable also to the classical stochastic differentials of completely positive processes, corresponding to the commutative Itô algebras, which are decomposable into the Wiener, Poisson and Newton orthogonal components.

2. QUANTUM FILTERING DYNAMICS

The quantum filtering theory, which was outlined in [1, 2] and developed then since [3], provides the derivations for new types of irreversible stochastic equations for quantum states, giving the dynamical solution for the well-known quantum measurement problem. Some particular types of such equations have been considered recently in the phenomenological theories of quantum permanent reduction [4, 5], continuous measurement collapse [6, 7], spontaneous jumps [8, 9], diffusions and localizations [10, 11]. The main feature of such dynamics is that the reduced irreversible evolution can be described in terms of a linear dissipative stochastic wave equation, the solution to which is normalised only in the mean square sense.

The simplest dynamics of this kind is described by the continuous filtering wave propagators $V_t(\omega)$, defined on the space Ω of all Brownian trajectories as an adapted operator-valued stochastic process in the system Hilbert space \mathcal{H} , satisfying the stochastic diffusion equation

$$(2.1) \quad dV_t + KV_t dt = LV_t dQ, \quad V_0 = I$$

in the Itô sense, which was derived from a unitary evolution in [13]. Here $Q(t, \omega)$ is the standard Wiener process, which is described by the independent increments $dQ(t) = Q(t+dt) - Q(t)$, having the zero mean values $\langle dQ \rangle = 0$ and the multiplication property $(dQ)^2 = dt$, K is an accretive operator, $K + K^\dagger \geq L^\dagger L$, and L is a linear operator $\mathcal{D} \rightarrow \mathcal{H}$. Using the Itô formula

$$(2.2) \quad d(V_t^\dagger V_t) = dV_t^\dagger V_t + V_t^\dagger dV_t + dV_t^\dagger dV_t,$$

and averaging $\langle \cdot \rangle$ over the trajectories of Q , one obtains $d\langle V_t^\dagger V_t \rangle \leq 0$ as a consequence of $L^\dagger L \leq K + K^\dagger$. Note that the process V_t is necessarily unitary if the filtering condition $K^\dagger + K = L^\dagger L$ holds, and if $L^\dagger = -L$ in the bounded case.

Another type of the filtering wave propagator $V_t(\omega) : \psi_0 \in \mathcal{H} \mapsto \psi_t(\omega)$ in \mathcal{H} is given by the stochastic jump equation

$$(2.3) \quad dV_t + KV_t dt = LV_t dP, \quad V_0 = I,$$

derived in [12] by the conditioning with respect to the spontaneous stationary reductions at the random time instants $\omega = \{t_1, t_2, \dots\}$. Here $L = J - I$ is the jump operator, corresponding to the stationary discontinuous evolutions $J : \psi_t \mapsto \psi_{t+}$ at

$t \in \omega$, and $P(t, \omega)$ is the standard Poisson process, counting the number $|\omega \cap [0, t]|$ compensated by its mean value t . It is described as the process with independent increments $dP(t) = P(t+dt) - P(t)$, having the values $\{0, 1\}$ at $dt \rightarrow 0$, with zero mean $\langle dP \rangle = 0$, and the multiplication property $(dP)^2 = dP + dt$. Using the Itô formula (2.2) with $dV_t^\dagger dV_t = V_t^\dagger L^\dagger L V_t (dP + dt)$, one can obtain

$$d(V_t^\dagger V_t) = V_t^\dagger (L^\dagger L - K - K^\dagger) V_t dt + V_t^\dagger (L^\dagger + L + L^\dagger L) V_t dP.$$

Averaging $\langle \cdot \rangle$ over the trajectories of P , one can easily find that $d\langle V_t^\dagger V_t \rangle \leq 0$ under the sub-filtering condition $L^\dagger L \leq K + K^\dagger$. Such evolution is unitary if $L^\dagger L = K + K^\dagger$ and if the jumps are isometric, $J^\dagger J = I$.

This proves in both cases that the stochastic wave function $\psi_t(\omega) = V_t(\omega)\psi_0$ is not normalized for each ω , but it is normalized in the mean square sense to the probability $\langle \|\psi_t\|^2 \rangle \leq \|\psi_0\|^2 = 1$ for the quantum system not to be demolished during its observation up to the time t . If $\langle \|\psi_t\|^2 \rangle = 1$, then the positive stochastic function $\|\psi_t(\omega)\|^2$ is the probability density of a diffusive \hat{Q} or counting \hat{P} output process up to the given t with respect to the standard Wiener Q or Poisson P input processes.

Using the Itô formula for $\phi_t(B) = V_t^\dagger B V_t$, one can obtain the stochastic equations

$$(2.4) \quad d\phi_t(B) + \phi_t(K^\dagger B + BK - L^\dagger B L) dt = \phi_t(L^\dagger B + B L) dQ,$$

$$(2.5) \quad d\phi_t(B) + \phi_t(K^\dagger B + BK - L^\dagger B L) dt = \phi_t(J^\dagger B J - B) dP,$$

describing the stochastic evolution $Y_t = \phi_t(B)$ of a bounded system operator $B \in \mathcal{L}(\mathcal{H})$ as $Y_t(\omega) = V_t(\omega)^\dagger B V_t(\omega)$. The maps $\phi_t : B \mapsto Y_t$ are Hermitian in the sense that $Y_t^\dagger = Y_t$ if $B^\dagger = B$, but in contrast to the usual Hamiltonian dynamics, are not multiplicative in general, $\phi_t(B^\dagger C) \neq \phi_t(B)^\dagger \phi_t(C)$, even if they are not averaged with respect to ω . Moreover, they are usually not normalized, $M_t(\omega) := \phi_t(\omega, I) \neq I$, although the stochastic positive operators $M_t = V_t^\dagger V_t$ under the filtering condition are usually normalized in the mean, $\langle M_t \rangle = I$, and satisfy the martingale property $\epsilon_t[M_s] = M_t$ for all $s > t$, where ϵ_t is the conditional expectation with respect to the history of the processes P or Q up to time t .

Although the filtering equations (2.3), (2.1) look very different, they can be unified in the form of quantum stochastic equation

$$(2.6) \quad dV_t + K V_t dt + K^- V_t d\Lambda_- = (J - I) V_t d\Lambda + L_+ V_t d\Lambda^+$$

where $\Lambda^+(t)$ is the creation process, corresponding to the annihilation $\Lambda_-(t)$ on the interval $[0, t]$, and $\Lambda(t)$ is the number of quanta on this interval. These canonical quantum stochastic processes, representing the quantum noise with respect to the vacuum state $|0\rangle$ of the Fock space \mathcal{F} over the single-quantum Hilbert space $L^2(\mathbb{R}_+)$ of square-integrable functions of $t \in [0, \infty)$, are formally given in [14] by the integrals

$$\Lambda_-(t) = \int_0^t \Lambda_-^r dr, \quad \Lambda^+(t) = \int_0^t \Lambda_+^r dr, \quad \Lambda(t) = \int_0^t \Lambda^r dr,$$

where Λ_-^r, Λ_r^+ are the generalized quantum one-dimensional fields in \mathcal{F} , satisfying the canonical commutation relations

$$[\Lambda_-^r, \Lambda_s^+] = \delta(s - r) I, \quad [\Lambda_-^r, \Lambda_-^s] = 0 = [\Lambda_r^+, \Lambda_s^+].$$

They can be defined by the independent increments with

$$(2.7) \quad \langle 0 | d\Lambda_- | 0 \rangle = 0, \quad \langle 0 | d\Lambda^+ | 0 \rangle = 0, \quad \langle 0 | d\Lambda | 0 \rangle = 0$$

and the noncommutative multiplication table

$$(2.8) \quad d\Lambda d\Lambda = d\Lambda, \quad d\Lambda_- d\Lambda = d\Lambda_-, \quad d\Lambda d\Lambda^+ = d\Lambda^+, \quad d\Lambda_- d\Lambda^+ = dt I$$

with all other products being zero: $d\Lambda d\Lambda_- = d\Lambda^+ d\Lambda = d\Lambda^+ d\Lambda_- = 0$. The standard Poisson process P as well as the Wiener process Q can be represented in \mathfrak{F} by the linear combinations [16]

$$(2.9) \quad P(t) = \Lambda(t) + i(\Lambda^+(t) - \Lambda_-(t)), \quad Q(t) = \Lambda^+(t) + \Lambda_-(t),$$

so the equation (2.6) corresponds to the stochastic diffusion equation (2.1) if $J = I$, $L_+ = L = -K^-$, and it corresponds to the stochastic jump equation (2.3) if $J = I + L$, $L_+ = iL = K^-$. The quantum stochastic equation for $\phi_t(B) = V_t^\dagger B V_t$ has the following general form

$$(2.10) \quad d\phi_t(B) + \phi_t(K^\dagger B + BK - L^- B L_+) dt = \phi_t(J^\dagger B J - B) d\Lambda \\ + \phi_t(J^\dagger B L_+ - K_+ B) + \phi_t(L^- B J - B K^-) d\Lambda_-,$$

where $L^- = L_+^\dagger$, $K_+^\dagger = K^-$, coinciding with either (2.4) or with (2.5) in the particular cases. The equation (2.10) is obtained from (2.6) by using the Itô formula (2.2) with the multiplication table (2.8). The sub-filtering condition $K + K^\dagger \leq L^- L_+$ for the equation (2.6) defines in both cases the positive operator-valued process $R_t = \phi_t(I)$ as a sub-martingale with $R_0 = I$, or a martingale in the case $K + K^\dagger = L^- L_+$. In the particular case

$$J = S, \quad K^- = L^- S, \quad L_+ = S K_+, \quad S^\dagger S = I,$$

corresponding to the Hudson–Evans flow if $S^\dagger = S^{-1}$, the evolution is isometric, and identity preserving, $\phi_t(I) = I$ in the case of bounded K and L .

In the next sections we define a multidimensional analog of the quantum stochastic equation (2.10) and will show that the suggested general structure of its generator indeed follows just from the property of complete positivity of the map ϕ_t for all $t > 0$ and the normalization condition $\phi_t(I) = M_t$ to a form-valued sub-martingale with respect to the natural filtration of the quantum noise in the Fock space \mathfrak{F} .

3. THE GENERATORS OF QUANTUM FILTERING COCYCLES.

The quantum filtering dynamics over an operator algebra $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ is described by a one parameter cocycles: $\phi = (\phi_t)_{t \geq 0}$ of linear completely positive stochastic maps $\phi_t(\omega) : \mathcal{B} \rightarrow \mathcal{B}$. The cocycle condition

$$(3.1) \quad \phi_s(\omega) \circ \phi_r(\omega^s) = \phi_{r+s}(\omega), \quad \forall r, s > 0$$

means the stationarity, with respect to the shift $\omega^s = \{\omega(t + s)\}$ of a given stochastic process $\omega = \{\omega(t)\}$. Such maps are in general unbounded, but normalized,

$\phi_t(I) = M_t$ to an operator-valued martingale $M_t = \epsilon_t[M_s] \geq 0$ with $M_0 = 1$, or a positive submartingale: $M_t \geq \epsilon_t[M_s]$, for all $s > t$, .

Now we give a noncommutative generalization of the quantum stochastic CP cocycles, which was suggested in [15] even for the nonlinear case. The stochastically differentiable family ϕ with respect to a quantum stationary process, with independent increments $\Lambda^s(t) = \Lambda(t+s) - \Lambda(s)$ generated by a finite dimensional Itô algebra is described by the quantum stochastic equation

$$(3.2) \quad d\phi_t(Y) = \phi_t \circ \lambda_\nu^\mu(Y) d\Lambda_\mu^\nu := \sum_{\mu, \nu} \phi_t(\lambda_\nu^\mu(Y)) d\Lambda_\mu^\nu, \quad Y \in \mathcal{B}$$

with the initial condition $\phi_0(Y) = Y$, for all $Y \in \mathcal{B}$. Here $\Lambda_\mu^\nu(t)$ with $\mu \in \{-, 1, \dots, d\}$, $\nu \in \{+, 1, \dots, d\}$ are the standard time $\Lambda_-^+(t) = tI$, annihilation $\Lambda_-^-(t)$, creation $\Lambda_+^+(t)$ and exchange-number $\Lambda_n^m(t) = N_n^m(t)$ operator integrators with $m, n \in \{1, \dots, d\}$. The infinitesimal increments $d\Lambda_\mu^\nu(t) = \Lambda_\mu^\nu(dt)$ are formally defined by the Hudson-Parthasarathy multiplication table [16] and the \flat -property [3],

$$(3.3) \quad d\Lambda_\mu^\beta d\Lambda_\gamma^\nu = \delta_\gamma^\beta d\Lambda_\mu^\nu, \quad \Lambda^\flat = \Lambda,$$

where δ_γ^β is the usual Kronecker delta restricted to the indices $\beta \in \{-, 1, \dots, d\}$, $\gamma \in \{+, 1, \dots, d\}$ and $\Lambda_{-\nu}^{b\mu} = \Lambda_{-\mu}^{\nu\dagger}$ with respect to the reflection $-(-) = +$, $-(+) = -$ of the indices $(-, +)$ only. The linear maps $\lambda_\nu^\mu : \mathcal{B} \rightarrow \mathcal{B}$ for the $*$ -cocycles $\phi_t^* = \phi_t$, where $\phi_t^*(Y) = \phi_t(Y^\dagger)^\dagger$, should obviously satisfy the \flat -property $\lambda^\flat = \lambda$, where $\lambda_{-\mu}^{b\nu} = \lambda_{-\nu}^{\mu*}$, $\lambda_\nu^{\mu*}(Y) = \lambda_\nu^\mu(Y^\dagger)^\dagger$. If the coefficients $b_\nu^\mu = \lambda_\nu^\mu(Y)$ are independent of t , ϕ satisfies the cocycle property $\phi_s \circ \phi_r^s = \phi_{s+r}$, where ϕ_t^s is the solution to (3.2) with $\Lambda_\nu^\mu(t)$ replaced by $\Lambda_\nu^{s\mu}(t)$. Define the $(d+2) \times (d+2)$ matrix $\mathbf{a} = [a_\nu^\mu]$ also for $\mu = +$ and $\nu = -$, by

$$\lambda_\nu^+(Y) = 0 = \lambda_-^\mu(Y), \quad \forall Y \in \mathcal{B},$$

and then one can extend the summation in (3.2) so it is also over $\mu = +$, and $\nu = -$. By such an extension the multiplication table for $d\Lambda(\mathbf{a}) = a_\nu^\mu d\Lambda_\mu^\nu$ can be written as

$$(3.4) \quad d\Lambda(\mathbf{a})^\dagger d\Lambda(\mathbf{a}) = d\Lambda(\mathbf{a}^\flat \mathbf{a})$$

in terms of the usual matrix product $(\mathbf{ba})_\nu^\mu = b_\lambda^\mu a_\nu^\lambda$ and the involution $\mathbf{a} \mapsto \mathbf{a}^\flat = \mathbf{b}, \mathbf{b}^\flat = \mathbf{a}$ can be obtained by the pseudo-Hermitian conjugation $a_\beta^{b\nu} = g_{\beta\mu} a_\gamma^{\mu\dagger} g^{\gamma\nu}$ respectively to the indefinite Minkowski metric tensor $g = [g_{\mu\nu}]$ and its inverse $g^{-1} = [g^{\mu\nu}]$, given by $g^{\mu\nu} = \delta_{-\nu}^\mu I = g_{\mu\nu}$.

Let us prove that the "spatial" part $\gamma = (\gamma_\nu^\mu)_{\nu \neq -}^{\mu \neq +}$ of $\gamma = \lambda + \delta$, called the quantum stochastic germ for the representation $\delta : B \mapsto (B\delta_\nu^\mu)_{\nu \neq -}^{\mu \neq +}$, must be completely stochastically dissipative for a CP cocycle ϕ in the following sense.

Theorem 1. *Suppose that the quantum stochastic equation (3.2) with $\phi_0(B) = B$ has a CP solution $\phi_t, t > 0$. Then the germ-map $\gamma = (\lambda_\nu^\mu + \delta_\nu^\mu)_{\nu=+, \bullet}^{\mu=-, \bullet}$ is conditionally completely positive*

$$\sum_k \iota(B_k) \eta_k = 0 \Rightarrow \sum_{k,l} \langle \eta_k | \gamma(B_k^\dagger B_l) | \eta_l \rangle \geq 0$$

Here $\boldsymbol{\eta} \in \mathcal{H} \oplus \mathcal{H}^\bullet$, $\mathcal{H}^\bullet = \mathcal{H} \otimes \mathbb{C}^d$, and $\boldsymbol{\iota} = (\iota_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,-,\bullet}$ is the degenerate representation $\iota_\nu^\mu(B) = B\delta_\nu^+\delta_-^\mu$, written both with γ in the matrix form as

$$(3.5) \quad \gamma = \begin{pmatrix} \gamma & \gamma_\bullet \\ \gamma_\bullet & \gamma_\bullet^\bullet \end{pmatrix}, \quad \boldsymbol{\iota}(B) = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},$$

where $\gamma = \lambda_+^-$, $\gamma^m = \lambda_+^m$, $\gamma_n = \lambda_n^-$, $\gamma_n^m = \delta_n^m + \lambda_n^m$ with $\delta_n^m(B) = B\delta_n^m$ such that

$$(3.6) \quad \gamma(B^\dagger) = \gamma(B)^\dagger, \quad \gamma^n(B^\dagger) = \gamma_n(B)^\dagger, \quad \gamma_n^m(B^\dagger) = \gamma_n^m(B)^\dagger$$

Proof. Let us denote by \mathcal{D} the \mathcal{H} -span $\left\{ \sum_f \xi^f \otimes f^\otimes \mid \xi^f \in \mathcal{H}, f^\bullet \in \mathbb{C}^d \otimes L^2(\mathbb{R}_+) \right\}$ of coherent (exponential) functions $f^\otimes(\tau) = \bigotimes_{t \in \tau} f^\bullet(t)$, given for each finite subset $\tau = \{t_1, \dots, t_n\} \subseteq \mathbb{R}_+$ by tensor products $f^{n_1, \dots, n_N}(\tau) = f^{n_1}(t_1) \dots f^{n_N}(t_N)$, where $f^n, n = 1, \dots, d$ are square-integrable complex functions on \mathbb{R}_+ and $\xi^f = 0$ for almost all $f^\bullet = (f^n)$. The co-isometric shift T_s intertwining $A^s(t)$ with $A(t) = T_s A^s(t) T_s^\dagger$ is defined on \mathcal{D} by $T_s(\eta \otimes f^\otimes)(\tau) = \eta \otimes f^\otimes(\tau + s)$. The complete positivity of the quantum stochastic adapted map ϕ_t into the \mathcal{D} -forms $\langle \chi \mid \phi_t(B) \psi \rangle$, for $\chi, \psi \in \mathcal{D}$ can be obviously written as

$$(3.7) \quad \sum_{X,Z} \sum_{f,h} \left\langle \xi_X^f \mid \phi_t(f^\bullet, X^\dagger Z, h^\bullet) \xi_Z^h \right\rangle \geq 0,$$

where

$$\langle \eta \mid \phi_t(f^\bullet, B, h^\bullet) \eta \rangle = \langle \eta \otimes f^\otimes \mid \phi_t(B) \eta \otimes h^\otimes \rangle e^{-\int_t^\infty f^\bullet(s)^\dagger h^\bullet(s) ds},$$

$\xi_B^f \neq 0$ for a finite sequence of $B_k \in \mathcal{B}$, and for a finite sequence of $f_l^\bullet = (f_l^1, \dots, f_l^d)$. If the \mathcal{D} -form $\phi_t(B)$ satisfies the stochastic equation (3.2), the \mathcal{H} -form $\phi_t(f^\bullet, B, h^\bullet)$ satisfies the differential equation

$$(3.8) \quad \begin{aligned} \frac{d}{dt} \phi_t(f^\bullet, B, h^\bullet) &= f^\bullet(t)^\dagger h^\bullet(t) \phi_t(f^\bullet, B, h^\bullet) + \phi_t(f^\bullet, \lambda_+^-(B), h^\bullet) \\ &+ \sum_{m=1}^d f^m(t)^* \phi_t(f^\bullet, \lambda_+^m(B), h^\bullet) + \sum_{n=1}^d h^n(t) \phi_t(f^\bullet, \lambda_n^-(B), h^\bullet) \\ &+ \sum_{m,n=1}^d f^m(t)^* h^n(t) \phi_t(f^\bullet, \lambda_n^m(B), h^\bullet), \end{aligned}$$

where $f^\bullet(t)^\dagger h^\bullet(t) = \sum_{n=1}^d f^n(t)^* h^n(t)$. The positive definiteness, (3.7), ensures the conditional positivity

$$(3.9) \quad \sum_f \sum_B B \xi_B^f = 0 \Rightarrow \sum_{X,Z} \sum_{f,h} \left\langle \xi_X^f \mid \gamma(f^\bullet, X^\dagger Z, h^\bullet) \xi_Z^h \right\rangle \geq 0$$

of the form $\gamma_t(f^\bullet, B, h^\bullet) = \frac{1}{t} (\phi_t(f^\bullet, B, h^\bullet) - B)$ for each $t > 0$ and of the limit γ_0 at $t \downarrow 0$, coinciding with the quadratic form

$$(3.10) \quad \left. \frac{d}{dt} \phi_t(f^\bullet, B, h^\bullet) \right|_{t=0} = \sum_{m,n} \bar{a}^m \gamma_n^m(B) c^n + \sum_m \bar{a}^m \gamma^m(B) + \sum_n \gamma_n(B) c^n + \gamma(B),$$

where $a^\bullet = f^\bullet(0)$, $c^\bullet = h^\bullet(0)$, and the γ 's are defined in (3.5). Hence the form

$$\begin{aligned} & \sum_{X,Z} \sum_{\mu,\nu} \langle \eta_X^\mu | \gamma_\nu^\mu (X^\dagger Z) \eta_Z^\nu \rangle := \sum_{X,Z} \sum_{m,n} \langle \eta_X^m | \gamma_n^m (X^\dagger Z) \eta_Z^n \rangle \\ & + \sum_{X,Z} \left(\sum_n \langle \eta_X | \gamma_n (X^\dagger Z) \eta_Z^n \rangle + \sum_m \langle \eta_X^m | \gamma^m (X^\dagger Z) \eta_Z \rangle + \langle \eta_X | \gamma (X^\dagger Z) | \eta_Z \rangle \right) \end{aligned}$$

with $\eta = \sum_f \xi^f$, $\eta^\bullet = \sum_f \xi^f \otimes a_f^\bullet$, where $a_f^\bullet = f^\bullet(0)$, is positive if $\sum_B B \eta_B = 0$. The components η and η^\bullet of these vectors are independent because for any $\eta \in \mathcal{H}$ and $\eta^\bullet = (\eta^1, \dots, \eta^d) \in \mathcal{H} \otimes \mathbb{C}^d$ there exists such a function $a^\bullet \mapsto \xi^a$ on \mathbb{C}^d with a finite support, that $\sum_a \xi^a = \eta$, $\sum_a \xi^a \otimes a^\bullet = \eta^\bullet$, namely, $\xi^a = 0$ for all $a^\bullet \in \mathbb{C}^d$ except $a^\bullet = 0$, for which $\xi^a = \eta - \sum_{n=1}^d \eta^n$ and $a^\bullet = e_n^\bullet$, the n -th basis element in \mathbb{C}^d , for which $\xi^a = \eta^n$. This proves the complete positivity of the matrix form γ , with respect to the matrix representation ι defined in (3.5) on the ket-vectors $\eta = (\eta^\mu)$. ■

4. A DILATION THEOREM FOR THE FORM-GENERATOR.

The conditional positivity of the structural map γ with respect to the degenerate representation ι written in the matrix form (3.6) obviously implies the positivity of the dissipation form

$$(4.1) \quad \sum_{X,Z} \langle \eta_X | \Delta(X, Z) \eta_Z \rangle := \sum_{k,l} \sum_{\mu,\nu} \langle \eta_k^\mu | \Delta_\nu^\mu(B_k, B_l) \eta_l^\nu \rangle,$$

where $\eta^- = \eta = \eta^+$ and $\eta_k = \eta_{B_k}$ for any (finite) sequence $B_k \in \mathcal{B}$, $k = 1, 2, \dots$, corresponding to non-zero $\eta_B = \eta_B \oplus \eta_B^\bullet$, $\eta_B \in \mathcal{H}$, $\eta_B^\bullet \in \mathcal{H}^\bullet$. Here $\Delta = (\Delta_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$ is the dissipator matrix,

$$\Delta(X, Z) = \gamma(X^\dagger Z) - \iota(X)^\dagger \gamma(Z) - \gamma(X)^\dagger \iota(Z) + \iota(X)^\dagger \gamma(I) \iota(Z),$$

given by the elements

$$\begin{aligned} (4.2) \quad \Delta_n^m(X, Z) &= \lambda_n^m(X^\dagger Z) + X^\dagger Z \delta_n^m, \\ \Delta_n^-(X, Z) &= \lambda_n^-(X^\dagger Z) - X^\dagger \lambda_n^-(Z) = \Delta_+^n(Z, X)^\dagger \\ \Delta_+^-(X, Z) &= \lambda_+^-(X^\dagger Z) - X^\dagger \lambda_+^-(Z) - \lambda_+^-(X^\dagger) Z + X^\dagger D Z, \end{aligned}$$

where $D = \lambda_+^-(I) \leq 0$ ($D = 0$ for the case of the martingale M_t). This means that the matrix-valued map $\gamma_\bullet^\bullet = [\gamma_n^m]$, is completely positive, and as follows from the next theorem, at least for the algebra $\mathcal{B} = \mathcal{B}(\mathcal{H})$ the maps γ , γ^m , γ_n have the following form

$$(4.3) \quad \begin{aligned} \gamma^m(B) &= \varphi^m(B) - K_m^\dagger B, & \gamma_n(B) &= \varphi_n(B) - B K_n \\ \gamma(B) &= \varphi(B) - K^\dagger B - B K, & \varphi(I) &\leq K + K^\dagger \end{aligned}$$

where $\varphi = (\varphi_\nu^\mu)_{\nu \neq -}^{\mu \neq +}$ is a completely positive bounded map from \mathcal{B} into the matrices of operators with the elements $\varphi_n^m = \gamma_n^m$, $\varphi_+^m = \varphi^m$, $\varphi_n^- = \varphi_n$, $\varphi_+^- = \varphi : \mathcal{B} \rightarrow \mathcal{B}$.

In order to make the formulation of the dilation theorem as concise as possible, we need the notion of the \flat -representation of the algebra \mathcal{B} in the operator algebra $\mathcal{A}(\mathcal{E})$ of a pseudo-Hilbert space $\mathcal{E} = \mathcal{H} \oplus \mathcal{H}^\circ \oplus \mathcal{H}$ with respect to the indefinite metric

$$(4.4) \quad (\xi | \xi) = 2 \operatorname{Re}(\xi^- | \xi^+) + \|\xi^\circ\|^2 + \|\xi^+\|_D^2$$

for the triples $\xi = (\xi^\mu)^{\mu=-,\circ,+} \in \mathcal{E}$, where $\xi^-, \xi^+ \in \mathcal{H}$, $\xi^\circ \in \mathcal{H}^\circ$, \mathcal{H}° is a pre-Hilbert space, and $\|\eta\|_D^2 = \langle \eta | D \eta \rangle$. The operators $A \in \mathcal{A}(\mathcal{E})$ are given by 3×3 -block-matrices $[A_\nu^\mu]_{\nu=-,\circ,+}^{\mu=-,\circ,+}$, having the Pseudo-Hermitian adjoints $(\xi | A^\flat \xi) = (A \xi | \xi)$, which are defined by the Hermitian adjoints $A_\nu^\mu = A_\mu^{\nu\dagger}$ as $A^\flat = G^{-1} A^\dagger G$ respectively to the indefinite metric tensor $G = [G_{\mu\nu}]$ and its inverse $G^{-1} = [G^{\mu\nu}]$, given by

$$(4.5) \quad \mathbf{G} = \begin{bmatrix} 0 & 0 & I \\ 0 & I_\circ^\circ & 0 \\ I & 0 & D \end{bmatrix}, \quad \mathbf{G}^{-1} = \begin{bmatrix} -D & 0 & I \\ 0 & I_\circ^\circ & 0 \\ I & 0 & 0 \end{bmatrix}$$

with an arbitrary D , where I_\circ° is the identity operator in \mathcal{H}° , being equal $I_\bullet^\bullet = [I \delta_n^m]_{n=1,\dots,d}^{m=1,\dots,d}$ in the case of $\mathcal{H}^\circ = \mathcal{H} \otimes \mathbb{C}^d = \mathcal{H}^\bullet$.

Theorem 2. *The following are equivalent:*

- (i) *The dissipation form (4.1), defined by the \flat -map λ with $\lambda_+^-(I) = D$, is positive definite: $\sum_{X,Z} \langle \eta_X | \Delta(X,Z) \eta_Z \rangle \geq 0$.*
- (ii) *There exists a pre-Hilbert space \mathcal{H}° , a unital \flat -representation j of \mathcal{B} in $\mathcal{B}(\mathcal{H}^\circ)$,*

$$(4.6) \quad j(B^\dagger B) = j(B)^\dagger j(B), \quad j(I) = I,$$

a (j, i) -derivation of \mathcal{B} with $i(B) = B$,

$$(4.7) \quad k(B^\dagger B) = j(B)^\dagger k(B) + k(B^\dagger) B,$$

having values in the operators $\mathcal{H} \rightarrow \mathcal{H}^\circ$, the adjoint map $k^(B) = k(B^\dagger)^\dagger$, with the property*

$$k^*(B^\dagger B) = B^\dagger k^*(B) + k^*(B^\dagger) j(B)$$

of (i, j) -derivation in the operators $\mathcal{H}^\circ \rightarrow \mathcal{H}$, and a map $l : \mathcal{B} \rightarrow \mathcal{B}$ having the coboundary property

$$(4.8) \quad l(B^\dagger B) = B^\dagger l(B) + l(B^\dagger) B + k^*(B^\dagger) k(B),$$

with the adjoint $l^(B) = l(B) + [D, B]$, such that $\gamma(B) = l(B) + DB$,*

$$\gamma_n(B^\dagger) = k(B)^\dagger L_n^\circ + B^\dagger L_n^- = \gamma^n(B)^\dagger,$$

and $\gamma_n^m(B) = L_m^{\circ\dagger} j(B) L_n^\circ$ for some operators $L_n^\circ : \mathcal{H} \rightarrow \mathcal{H}^\circ$ having the adjoints $L_n^{\circ\dagger}$ on \mathcal{H}° and $L_n^- \in \mathcal{B}$.

- (iii) *There exists a pseudo-Hilbert space, \mathcal{E} , a unital \flat -representation $j : \mathcal{B} \rightarrow \mathcal{A}(\mathcal{E})$, and a linear operator $\mathbf{L} : \mathcal{H} \oplus \mathcal{H}^\bullet \rightarrow \mathcal{E}$ such that*

$$(4.9) \quad \mathbf{L}^\flat j(B) \mathbf{L} = \gamma(B), \quad \forall B \in \mathcal{B}.$$

- (iv) *The structural map $\gamma = \lambda + \delta$ is conditionally completely positive with respect to the matrix representation ι in (3.5).*

Proof. The implication (i) \Rightarrow (ii) generalizes the Evans-Lewis Theorem[17], and its proof is similar to the proof of the dilation theorem in [18]. Let \mathcal{H}° be the pre-Hilbert space of Kolmogorov decomposition $\Delta(X, Z) = \mathbf{k}(X)^\dagger \mathbf{k}(Z)$. It is defined as the

quotient space $\mathcal{H}^\circ = \mathcal{K}/\mathcal{I}$ of the \mathcal{H} -span $\mathcal{K} = \{(\boldsymbol{\eta}_B)_{B \in \mathcal{B}}\}$, where $\boldsymbol{\eta}_B \in \mathcal{H} \oplus \mathcal{H}^\bullet$ is not equal zero only for a finite number of $B \in \mathcal{B}$, with respect to the kernel

$$\mathcal{I} = \left\{ (\boldsymbol{\eta}_B)_{B \in \mathcal{B}} \in \mathcal{K} \mid \sum_{X,Z} \langle \boldsymbol{\eta}_X \mid \boldsymbol{\Delta}(X,Z) \boldsymbol{\eta}_Z \rangle = 0 \right\}$$

of the positive-definite form (4.1). The operators $\mathbf{k}(B)^\dagger : \mathcal{H}^\circ \rightarrow \mathcal{H} \oplus \mathcal{H}^\bullet$ are defined on the classes $\boldsymbol{\eta}^\circ$ of $(\boldsymbol{\eta}_X)_{X \in \mathcal{B}} \in \mathcal{K}$ as the adjoint

$$\langle \mathbf{k}(B)^\dagger \boldsymbol{\eta}^\circ \mid \boldsymbol{\eta} \rangle = \sum_X \langle \boldsymbol{\eta}_X \mid \boldsymbol{\Delta}(X,B) \boldsymbol{\eta} \rangle$$

to the bounded operators $\mathbf{k}(B) : \mathcal{H} \oplus \mathcal{H}^\bullet \rightarrow \mathcal{H}^\circ$, mapping the pairs $\boldsymbol{\eta} = \eta \oplus \eta^\bullet$ into the equivalence classes $\boldsymbol{\eta}^\circ(B) = \mathbf{k}(B) \eta + k_\bullet(B) \eta^\bullet$ of $(\delta_Z(B) \boldsymbol{\eta})_{Z \in \mathcal{B}}$, where $\delta_Z(B) = 1$ if $B = Z$, otherwise $\delta_Z(B) = 0$. Let us define a linear operator $j(B)$ on \mathcal{H}° by

$$j(B) \sum_Z (k(Z) \eta + k_\bullet(Z) \eta^\bullet) = \sum_Z (k(BZ) \eta - k(B) Z \eta + k_\bullet(BZ) \eta^\bullet).$$

Obviously $j(XB) = j(X) j(B)$, $j(I) = I$ because $k(I) = 0$ and as follows from the definition of the dissipation form, $j(B)^\dagger = j(B^\dagger)$ for all $B \in \mathcal{B}$. Thus j is a unital \dagger -representation, k is a (j, i) -cocycle, and $k_\bullet(B) = j(B) L_\bullet^\circ$, where $L_\bullet^\circ = k_\bullet(I)$. Moreover, as

$$\begin{aligned} \gamma(B^\dagger B) + B^\dagger \gamma(I) B &= B^\dagger \gamma(B) + \gamma(B^\dagger) B + k(B)^\dagger k(B), \\ \gamma_\bullet(B^\dagger B) &= k_\bullet(B)^\dagger k_\bullet(B), \\ \gamma_\bullet(B^\dagger B) - B^\dagger \gamma_\bullet(B) &= k(B)^\dagger k_\bullet(B) = \gamma^\bullet(B^\dagger B)^\dagger - \gamma^\bullet(B)^\dagger B, \end{aligned}$$

the property (4.8) is fulfilled, $L_\bullet^\circ j(B) L_\bullet^\circ = \gamma_\bullet(B)$ with $L_\bullet^\circ = k_\bullet^*(I) = L_\bullet^{\circ\dagger}$, and

$$\gamma_\bullet(B^\dagger) = k(B)^\dagger L_\bullet^\circ + B^\dagger L_\bullet^- = \gamma^\bullet(B)^\dagger,$$

where $L_\bullet^- = \gamma_\bullet(I)$, $L_+^\bullet = \gamma^\bullet(I) = L_\bullet^{-\dagger}$.

The proof of the implication (ii) \Rightarrow (iii) can be also obtained as in [18] by the explicit construction of \mathcal{E} as $\mathcal{H} \oplus \mathcal{H}^\circ \oplus \mathcal{H}$ with the indefinite metric tensor $\mathbf{G} = [G_{\mu\nu}]$ given above for $\mu, \nu = -, \circ, +$, and $D = \gamma(I)$. The unital \mathfrak{b} -representation $j = [j_\nu^\mu]_{\nu=-,\circ,+}^{\mu=-,\circ,+}$ of \mathcal{B} on \mathcal{E} :

$$j(X^\dagger Z) = j(X)^\mathfrak{b} j(Z), \quad j(I) = I$$

with $j(B)^\mathfrak{b} = \mathbf{G}^{-1} j(B)^\dagger \mathbf{G} = j(B^\dagger)$ is given by the components

$$(4.10) \quad j_\circ^\circ = j, \quad j_+^\circ = k, \quad j_\circ^- = k^*, \quad j_+^- = l, \quad j_-^- = i = j_+^+$$

and all other $j_\nu^\mu = 0$. The linear operator $\mathbf{L} : \mathcal{H} \oplus \mathcal{H}^\bullet \rightarrow \mathcal{E}$, where $\mathcal{H}^\bullet = \mathcal{H} \otimes \mathbb{C}^d$, can be defined by the components (L^μ, L_\bullet^μ) ,

$$L^- = 0, \quad L^\circ = 0, \quad L^+ = I, \quad L_\bullet^- = (L_n^-), \quad L_\bullet^\circ = (L_n^\circ), \quad L_\bullet^+ = 0,$$

and $\mathbf{L}^\flat = \begin{pmatrix} I & 0 & D \\ 0 & L_\circ^\bullet & L_+^\bullet \end{pmatrix} = \mathbf{L}^\dagger \mathbf{G}$, where $L_\circ^\bullet = L_\circ^{\circ\dagger}$, $L_+^\bullet = L_\bullet^{-\dagger}$. Then $\mathbf{L}^\flat \mathbf{J} \mathbf{L} =$

$$\begin{pmatrix} 0 & L_\bullet^- \\ 0 & L_\circ^\bullet \\ 1 & 0 \end{pmatrix}^\flat \begin{bmatrix} i & k^* & l \\ 0 & j & k \\ 0 & 0 & i \end{bmatrix} \begin{pmatrix} 0 & L_\bullet^- \\ 0 & L_\circ^\bullet \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} l + Di & k^* L_\circ^\bullet + i L_\bullet^- \\ L_\circ^\bullet k + L_+^\bullet i & L_\circ^\bullet j L_\circ^\bullet \end{pmatrix} = \gamma$$

In order to prove the implication (iii) \Rightarrow (iv), it is sufficient to show that the vectors $\xi = \sum_B j(B) \mathbf{L} \eta_B$ are positive, $(\xi|\xi) \geq 0$ if $\sum_B \iota(B) \eta_B = \sum_B B \eta_B = 0$. But this follows immediately from the observation $\xi^+ = \sum_B j(B) L^+ \eta_B = \sum_B B \eta_B = 0$ such that the indefinite metrics (4.4) is positive, $(\xi|\xi) = \|\xi^\circ\|^2 \geq 0$ in this case.

The final implication (iv) \Rightarrow (i) is obtained as the case $\eta_I = -\sum_{B \neq I} B \eta_B$ of $\sum_B B \eta_B = 0$. ■

5. THE STRUCTURE OF THE BOUNDED FILTERING GENERATORS.

The structure (4.3) of the form-generator for CP cocycles over $\mathcal{B} = \mathcal{B}(\mathcal{H})$ is a consequence of the well known fact that the derivations k, k^* of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a Hilbert space \mathcal{H} are spatial, $k(B) = j(B) L - LB$, $k^*(B) = L^\dagger j(B) - BL^\dagger$, and so

$$(5.1) \quad l(B) = \frac{1}{2} (L^\dagger k(B) + k^*(B) L + [B, D]) + i[H, B],$$

where $H^\dagger = H$ is a Hermitian operator in \mathcal{H} . The germ-map γ whose components are composed (as in (4.3)) into the sums of the components φ_ν^μ of a CP matrix map $\varphi : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{M}(\mathbb{C}^{d+1})$ and left and right multiplications, are obviously conditionally completely positive with respect to the representation ι in (4). As follows from the dilation theorem in this case, there exists a family $L_- = L = L_+$, $L_n = L_n^\circ$, $n = 1, \dots, d$ of linear operators $L_\nu : \mathcal{H} \rightarrow \mathcal{H}^\circ$, having adjoints $L_\mu^\dagger : \mathcal{H}^\circ \rightarrow \mathcal{H}$ such that $\varphi_\nu^\mu(B) = L_\mu^\dagger j(B) L_\nu$.

The next theorem proves that these structural conditions which are sufficient for complete positivity of the cocycles, given by the equation (3.2), are also necessary if the germ-map γ is w*-continuous on an operator algebra \mathcal{B} . Thus the equation (3.2) for a completely positive quantum cocycle with bounded stochastic derivatives has the following general form

$$(5.2) \quad \begin{aligned} d\phi_t(B) + \phi_t(K^\dagger B + BK - L^\dagger j(B) L) dt &= \sum_{m,n=1}^d \phi_t(L_m^\dagger j(B) L_n - B \delta_n^m) d\Lambda_m^n \\ &+ \sum_{m=1}^d \phi_t(L_m^\dagger j(B) L - K_m^\dagger B) d\Lambda_m^+ + \sum_{n=1}^d \phi_t(L^\dagger j(B) L_n - BK_n) d\Lambda_-^n, \end{aligned}$$

generalising the Lindblad form [17], for the norm-continuous semigroups of completely positive maps. The quantum stochastic submartingale $M_t = \phi_t(I)$ is defined by the integral

$$M_t + \int_0^t \phi_s(D) ds = I + \int_0^t \sum_{m,n} \phi_s(L_n^\dagger L_m - \delta_m^n) d\Lambda_n^m$$

$$(5.3) \quad + \int_0^t \sum_{m=1}^d \phi_s (L_m^\dagger L - K_m^\dagger) d\Lambda_m^+ + \int_0^t \sum_{n=1}^d \phi_s (L^\dagger L_n - K_n) d\Lambda_n^-.$$

If the space \mathcal{K} can be embedded into the direct sum $\mathcal{H} \otimes \mathbb{C}^d = \mathcal{H} \oplus \dots \oplus \mathcal{H}$ of d copies of the initial Hilbert space \mathcal{H} such that $j(B) = (B\delta_n^m)$, this equation can be resolved in the form $\phi_t(B) = F_t^\dagger B F_t$, where $F = (F_t)_{t \geq 0}$ is an (unbounded) cocycle in the tensor product $\mathcal{H} \otimes \mathcal{F}$ with Fock space \mathcal{F} over the Hilbert space $\mathbb{C}^d \otimes L^2(\mathbb{R}_+)$ of the quantum noise of dimensionality d . The cocycle F satisfies the quantum stochastic equation

$$(5.4) \quad dF_t + K F_t dt = \sum_{i,n=1}^d (L_n^i - I\delta_n^i) F_t d\Lambda_i^n + \sum_{i=1}^d L^i F_t d\Lambda_i^+ - \sum_{n=1}^d K_n F_t d\Lambda_n^-,$$

where L_n^i and L^i are the operators in \mathcal{H} , defining

$$(5.5) \quad \begin{aligned} \varphi_n^m(B) &= \sum_{i=1}^d L_m^{i\dagger} B L_n^i, & \varphi(B) &= \sum_{i=1}^d L^{i\dagger} B L^i \\ \varphi^m(B) &= \sum_{i=1}^d L_m^{i\dagger} B L^i, & \varphi_n(B) &= \sum_{i=1}^d L^{i\dagger} B L_n^i \end{aligned}$$

with $\sum_{i=1}^d L^{i\dagger} L^i = K + K^\dagger$ if M_t is a martingale ($\leq K + K^\dagger$ if submartingale).

Theorem 3. *Let the germ-maps γ of the quantum stochastic cocycle ϕ over a von-Neumann algebra \mathcal{B} be w^* -continuous and bounded:*

$$(5.6) \quad \|\gamma\| < \infty, \quad \|\gamma_\bullet\| = \left(\sum_{n=1}^d \|\gamma_n\|^2 \right)^{\frac{1}{2}} = \|\gamma^\bullet\| < \infty, \quad \|\gamma_\bullet^\bullet\| = \|\gamma_\bullet^\bullet(I)\| < \infty,$$

where $\|\gamma\| = \sup \{ \|\gamma(B)\| : \|B\| < 1 \}$, $\|\gamma_\bullet^\bullet(I)\| = \sup \{ \langle \eta^\bullet, \gamma_\bullet^\bullet(I) \eta^\bullet \rangle : \|\eta^\bullet\| < 1 \}$ and ϕ_t be a CP cocycle, satisfying equation (3.2) with $\phi_0(B) = B$ and normalized to a submartingale (martingale). Then they have the form (4.3) written as

$$(5.7) \quad \gamma(B) = \varphi(B) - \iota(B) K - K^\dagger \iota(B)$$

with $\varphi = \varphi_+^-$, $\varphi^m = \varphi_+^m$, $\varphi_n = \varphi_n^-$ and $\varphi_n^m = \gamma_n^m$, composing a bounded CP map.

$$(5.8) \quad \varphi = \begin{pmatrix} \varphi & \varphi_\bullet^\bullet \\ \varphi_\bullet & \varphi_\bullet^\bullet \end{pmatrix}, \quad \text{and} \quad K = \begin{pmatrix} K & K_\bullet^\bullet \\ K_\bullet & K_\bullet^\bullet \end{pmatrix}$$

with arbitrary $K^\bullet, K_\bullet^\bullet$, and $K + K^\dagger \geq \varphi(I)$. The equation (5.2) has the unique CP solution, satisfying the condition $\phi_s(I) \leq \epsilon_s[\phi_t(I)]$ for all $s < t$ ($\phi_s(I) = \epsilon_s[\phi_t(I)]$ if $K + K^\dagger = \varphi(I)$).

Proof. The structure (5.7) for the CP component γ_\bullet^\bullet was obtained as a part of the dilation theorem in the Stinespring form $\gamma_\bullet^\bullet(B) = L_\bullet^\dagger j(B) L_\bullet = \varphi_\bullet^\bullet(B)$, where $L_\bullet = L_\bullet^\circ$. In order to obtain the structure (5.7) for the bounded germ-maps γ_\bullet and γ^\bullet , we can take into account the spatial structure $k(B) = j(B)L - LB$ of a

bounded (j, i) -derivation for a von-Neumann algebra \mathcal{B} with respect to a normal representation j of \mathcal{B} and $i(B) = B$. Then

$$\gamma_{\bullet}(B) = k^*(B) L_{\bullet}^{\circ} + B L_{\bullet}^{-} = L^{\dagger} j(B) L_{\bullet}^{\circ} - B (L^{\dagger} L_{\bullet}^{\circ} - L_{\bullet}^{-}) = L_{+}^{\dagger} j(B) L_{\bullet} - B K_{\bullet},$$

where $L_{+} = L, K_{\bullet} = L^{\dagger} L_{\bullet}^{\circ} - L_{\bullet}^{-}$. Hence $\gamma_{\bullet}(B) = \varphi_{\bullet}(B) - B K_{\bullet}, \gamma^{\bullet}(B) = \varphi^{\bullet}(B) - K^{\bullet} B = \gamma_{\bullet}^*(B)$, where $K^{\bullet} = K_{\bullet}^{\dagger}, \varphi^{\bullet}(B) = L_{\bullet}^{\dagger} j(B) L = \varphi_{\bullet}^*(B)$, such that the matrix-map $\varphi(B) = (L^{\mu} j(B) L_{\nu})_{\nu=+, \bullet}^{\mu=-, \bullet}$ with $L^{-} = L^{\dagger}, L^{\bullet} = L_{\bullet}^{\dagger}$ is CP. Taking into account the form (5.1) of the coboundary $l(B) = \gamma(B) - DB$ which is due to the spatial form $[iH + \frac{1}{2}D, B]$ of the bounded derivation $l(B) - \frac{1}{2}(L^{\dagger} k(B) + k^*(B) L)$ on \mathcal{B} , one can obtain the representation

$$\gamma(B) = \frac{1}{2} (L^{\dagger} k(B) + k^*(B) L + DB + BD) + i[H, B] = \varphi(B) - BK - K^{\dagger} B,$$

where $\varphi(B) = L^{\dagger} j(B) L, K = iH + \frac{1}{2}(L^{\dagger} L - D)$.

The existence and uniqueness of the solutions $\phi_t(B)$ to the quantum stochastic equations (3.2) with the bounded generators $\lambda_{\nu}^{\mu}(B) = \gamma_{\nu}^{\mu}(B) - B\delta_{\nu}^{\mu}$ and the initial conditions $\phi_0(B) = B$ in an operator algebra \mathcal{B} was proved in [20]. The positivity of the solutions in the case of the equation (5.2), corresponding to the conditionally positive germ-function (5.7), can be obtained by the iteration

$$\phi_t^{(n+1)}(B) = V_t^{\dagger} B V_t + \int_0^t \phi_s^{(n)} \left(\beta_{\nu}^{\mu} \left(V_t^{\dagger}(s) B V_t(s) \right) \right) d\Lambda_{\mu}^{\nu}, \quad \phi_t^{(0)}(B) = B$$

of the quantum stochastic integral equation

$$(5.9) \quad \phi_t(B) = V_t^{\dagger} B V_t + \int_0^t \phi_s \left(\beta_{\nu}^{\mu} \left(V_t^{\dagger}(s) B V_t(s) \right) \right) d\Lambda_{\mu}^{\nu},$$

with $\beta_{\nu}^{\mu}(B) = \varphi_{\nu}^{\mu}(B) - B\delta_{\nu}^{\mu}$. Here $V_t = V_t(s) V_s$ with $V_t(s) = T_s^{\dagger} V_{t-s} T_s$ shifted by the co-isometry T_s in \mathcal{D} , is the vector cocycle, resolving the quantum stochastic differential equation

$$(5.10) \quad dV_t + K V_t dt + \sum_{m=1}^d K_m V_t d\Lambda_{-}^m = 0$$

with the initial condition $V_0 = I$ in \mathcal{H} . The equivalence of (5.2) and (5.9), (5.10) is verified by direct differentiation of (5.9). In order to prove the complete positivity of this solution, one should write down the corresponding iteration

$$\phi_t^{(n+1)}(f^{\bullet}, B, h^{\bullet}) = V_t^{\dagger} B V_t + \int_0^t f(s)^{\dagger} \phi_s^{(n)} \left(f^{\bullet}, \varphi \left(V_t^{\dagger}(s) B V_t(s) \right), h^{\bullet} \right) h(s) ds,$$

of the ordinary integral equation for the operator-valued kernels of coherent vectors, defined in (3.7). Here $g(s) = 1 \oplus g^{\bullet}(s)$ such that

$$\begin{aligned} \sum_{X,Z} \sum_{f,h} \left\langle \xi_X^f \middle| \phi_t(f^{\bullet}, X^{\dagger} Z, h^{\bullet}) \xi_Z^h \right\rangle &= \sum_{X,Z} \langle X V_t \eta_X | Z V_t \eta_Z \rangle \\ &+ \int_0^t \sum_{X,Z} \sum_{f,h} \left\langle \eta_X^f(s) \middle| \phi_s(f^{\bullet}, \varphi(X^{\dagger} Z), h^{\bullet}) \eta_Z^h(s) \right\rangle, \end{aligned}$$

where $\eta_B = \sum_g \xi_B^g, \eta_B^g(s) = \sum_{g(s)} \xi_B^g \otimes g(s)$. Then the CP property for $\phi_t^{(n)}$, immediately follows from the CP property of $\phi_s^{(n-1)}, s < t$ and of φ . The direct iteration of this integral recursion with the initial CP condition $\phi_t^{(0)}(B) = B$ gives

at the limit $n \rightarrow \infty$ the minimal CP solution in the form of sum of n-tupol CP integrals on the interval $[0, t]$. ■

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